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LETTER TO THE EDITOR

The O(3) gauge transformation and 3-state vertex models

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Abstract. We consider the O(3) gauge transformation for three-state vertex models on *lattices of coordination number three*. Using an explicit mapping between O(3) and SL(2), we establish that there exist exactly six polynomials of the vertex weights, which are fundamentally invariant under the O(3) transformation. Explicit expressions of these fundamental invariants are obtained in the case of symmetric vertex weights.

The consideration of gauge transformations has played a central role in the study of discrete spin systems. The gauge transformation is a linear transformation of the Boltzmann weights of a spin system, such as a vertex model, which does not alter the partition function. In a classic paper Wegner [1] formulated the gauge transformation for discrete spin systems, generalizing the previously known duality and weak-graph transformations. Properties pertaining to specific spin and lattice systems remain, however, to be worked out on a case by case basis. For example, those pertaining to the O(2) transformation for the 16-vertex model on the square lattice have subsequently been studied by Hijmans *et al* [2–4].

Of particular interest in statistical mechanics is the construction of invariants of the transformation, a subject matter of great interest in mathematics at the turn of the century [5–7]. In statistical mechanics the invariants of the O(2) transformation for 2-state vertex models have been utilized to determine the criticality of the Ising models in a non-zero magnetic field [8–12]. In the case of the O(2) transformation it has been possible to explicitly construct the invariants [12, 13]. The direct construction of invariants for O(3) is more complicated, however. But the day is saved since there exists a mapping between O(3) and SL(2), and invariants for the latter are already known. In this letter we utilize this mapping to obtain invariants of the O(3) gauge transformation which is applicable to 3-state spin systems.

Consider a lattice of coordination number 3, *which can be in any spatial dimension*, and assume that each of the lattice edges can be independently in one of three distinct states. With each lattice site we associate a vertex weight $W(s_1, s_2, s_3)$, where $s_i = 1, 2$ and 3 specifies the states of the three incident edges. This defines a 27-vertex model and the partition function $Z = \sum \prod W(s_1, s_2, s_3)$, where the summation is taken over all edge configurations of the lattice.

Wegner [1] has shown that the partition function Z remains unchanged if the vertex weights W are replaced by $\tilde{W}$ given by

$$\tilde{W}(t_1, t_2, t_3) = \sum_{s_1=1}^{3} \sum_{s_2=1}^{3} \sum_{s_3=1}^{3} R_{t_1 s_1} R_{t_2 s_2} R_{t_3 s_3} W(s_1, s_2, s_3) \tag{1}$$

provided that R_{ts} are elements of a 3×3 matrix $\mathbf{R}$ associated with lattice edges satisfying $\tilde{\mathbf{R}}\mathbf{R}=\mathbf{I}$, where $\tilde{\mathbf{R}}$ is the transpose of $\mathbf{R}$ and $\mathbf{I}$ is the identity matrix. This implies $\det|R_{ts}|=\pm 1$, and, consequently, the transformation (1) leaves $\sum_{s_1,s_2,s_3} W^2(s_1, s_2, s_3)$ invariant and thus gives rise to a representation of the three-dimensional orthogonal group O(3). In reality the validity of the invariance of the partition function holds more generally even if $\mathbf{R}$ is edge-dependent [1]. For this reason we refer to (1) as the O(3) gauge transformation.

Explicitly, O(3) is a three-parameter group. For SO(3) or $\det|R_{ts}|=1$, e.g., we can write

$$\mathbf{R}=\begin{pmatrix} c_2c_3 & -s_1s_2c_3+c_1s_3 & c_1s_2c_3+s_1s_3 \\ -c_2s_3 & c_1c_3+s_1s_2s_3 & -c_1s_2s_3+s_1c_3 \\ -s_2 & -s_1c_2 & c_1c_2 \end{pmatrix} \tag{2}$$

where $c_i=\cos\theta_i$, $s_i=\sin\theta_i$. This can be interpreted as a rotation in the 3-space by first making a rotation θ_1 about the x axis, followed by a rotation of θ_2 about the y axis and finally a rotation θ_3 about z axis [14].

Generally, the transformation (1) forms a representation of O(3) in the space of tensors of rank 3. Let y_1, y_2, y_3 be the coordinates of the fundamental representation space of O(3). Then the general tensors of rank 3 form a 3^3-dimensional space with basis $y_m\otimes y_n\otimes y_k$, where the three y's (first, second, and third) refer to specific incident edges, and the subscripts specify the state of the incident edge.

The consideration is much simplified when the vertex weights are symmetric, i.e. $W(s_1, s_2, s_3)$ is independent of the permutation of s_1, s_2, and s_3. In this case, we can conveniently relabel the vertex weights as ω_{ijk}, where i, j, k are, respectively, the numbers of incident edges in states 1, 2, 3 subject to $i+j+k=3$. Thus, the 27 vertex weights reduce to 10 independent ones whose associated configurations are shown in figure 1, and (1) gives rise to a 10×10 matrix representation of O(3). Furthermore, the tensor product of the basis $y_m\otimes y_n\otimes y_k$ can be replaced by an ordinary product, and the vertex weights can be written as given by the polynomial representation

$$\omega_{ijk}=y_1^i y_2^j y_3^k \qquad i+j+k=3. \tag{3}$$

It is well known that the special unitary group SU(2) is two-to-one homomorphic to SO(3), a familiar example being the spinor representation of the rotation group in

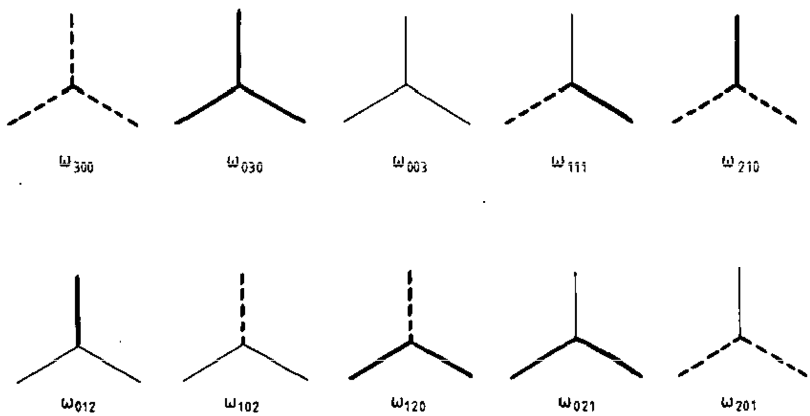

Figure 1. The ten vertex configurations and the weights of a symmetric 3-state 27-vertex model. The vertex configuration with weight ω_{ijk} is characterized by i broken, j thick, and k thin lines.

quantum mechanics. In addition, the invariants of SU(2) are identical to those of the special linear group SL(2). It follows that we can deduce the invariants of O(3) from those already known for SL(2). (Strictly speaking, this leads to invariants for SO(3), which may change sign under the odd elements of O(3).) We first describe the mapping of the representations for the two groups.

Let α_1 and α_2 be the coordinates of the fundamental representation space of SL(2). The mapping between α_1, α_2 and the coordinates y_1, y_2, y_3 of the vector representation of O(3) is

$$z_1 \equiv \alpha_1^2 = -y_1 + \mathrm{i}y_3 \qquad z_2 \equiv \alpha_1\alpha_2 = y_2 \qquad z_3 \equiv \alpha_2^2 = y_1 + \mathrm{i}y_3 \tag{4}$$

where z_1, z_2, z_3 form the coordinates of a rank-two symmetric tensor.

In view of (3) and (4), ω_{ijk} are raised to the sixth power of α_i and therefore invariants of O(3) must be given by tensors of rank six in $\{\alpha_1, \alpha_2\}$ with elements in the binary form

$$\begin{aligned}
&e_0 \equiv \alpha_1^6 = z_1^3 \qquad e_1 \equiv \alpha_1^5\alpha_2 = z_1^2 z_2\\
&e_2 \equiv \alpha_1^4\alpha_2^2 = (z_1^2 z_3 + 4z_1 z_2^2)/5\\
&e_3 \equiv \alpha_1^3\alpha_2^3 = (2z_2^3 + 3z_1 z_2 z_3)/5\\
&e_4 \equiv \alpha_1^2\alpha_2^4 = (z_3^2 z_1 + 4z_3 z_2^2)/5\\
&e_5 \equiv \alpha_1\alpha_2^5 = z_2 z_3^2 \qquad e_6 \equiv \alpha_2^6 = z_3^3.
\end{aligned} \tag{5}$$

Here, coefficients on the RHS are determined according to the following rules: (i) write each e_j as the average of all distinct permutations of the six α_i, (ii) for each permutation, group the six z_i into three consecutive pairs, and (iii) replace the grouped pairs by z_i using (4). For example, the first four lines of (5) are obtained from:

$$e_0 = (\alpha_1\alpha_1)(\alpha_1\alpha_1)(\alpha_1\alpha_1) = z_1^3$$

$$e_1 = \tfrac{1}{6}[(\alpha_1\alpha_1)(\alpha_1\alpha_1)(\alpha_1\alpha_2) + \text{all permutations of the six } \alpha_i] = \tfrac{1}{6}(6z_1^2 z_2) = z_1^2 z_2$$

$$e_2 = \tfrac{1}{15}[(\alpha_1\alpha_1)(\alpha_1\alpha_1)(\alpha_2\alpha_2) + \text{all permutations of the six } \alpha_i] = \tfrac{1}{15}(12z_1^2 z_2 + 3z_1^2 z_3)$$

$$e_3 = \tfrac{1}{20}[(\alpha_1\alpha_1)(\alpha_1\alpha_2)(\alpha_2\alpha_2) + \text{all permutations of the six } \alpha_i] = \tfrac{1}{20}(12z_1 z_2 z_3 + 8z_2^3).$$

The polynomial nature of symmetric tensors now makes it possible to simply substitute (4) and (3) into (5), leading to the following explicit expressions for the e_j:

$$\begin{aligned}
&e_0 = u + \mathrm{i}v \qquad e_1 = s + \mathrm{i}t \qquad e_2 = (x + \mathrm{i}y)/5\\
&e_6 = -u + \mathrm{i}v \qquad e_5 = s - \mathrm{i}t \qquad e_4 = (-x + \mathrm{i}y)/5\\
&e_3 = (2\omega_{030} - 3\omega_{210} - 3\omega_{012})/5
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
&u = 3\omega_{102} - \omega_{300} \qquad x = \omega_{300} + \omega_{102} - 4\omega_{120}\\
&v = 3\omega_{201} - \omega_{003} \qquad y = 4\omega_{021} - \omega_{003} - \omega_{201}\\
&a = \omega_{210} - \omega_{012} \qquad t = 2\omega_{111}.
\end{aligned} \tag{7}$$

We now look for polynomials of the vertex weights (3) which are the *fundamental* invariants of the O(3) transformation, i.e. they cannot be expressed as invariants of lower degrees and all other polynomial invariants are polynomials of them.

The ten-dimensional representation of O(3) can be decomposed into two invariant subspaces of dimensions 3 and 7. While group-theoretic argument exists for its reasoning, this decomposition also arises as a consequence of the mapping between O(3) and SL(2) in the binary form (5): the presence of seven elements in (5) implies the existence of a seven-dimensional invariant subspace.

The elements of three-dimensional invariant subspace can be found easily. They are

$$\begin{aligned}\eta_1 &= \omega_{300}+\omega_{102}+\omega_{120}=y_1(y_1^2+y_2^2+y_3^2)\\ \eta_2 &= \omega_{030}+\omega_{012}+\omega_{210}=y_2(y_1^2+y_2^2+y_3^2)\\ \eta_3 &= \omega_{003}+\omega_{021}+\omega_{201}=y_3(y_1^2+y_2^2+y_3^2).\end{aligned} \tag{8}$$

Obviously, this subspace transforms in the same way as the $\{y_1, y_2, y_3\}$ space. There is only one fundamental invariant in this subspace, namely,

$$I_0=\eta_1^2+\eta_2^2+\eta_3^2. \tag{9}$$

To find the fundamental invariants in the seven-dimensional invariant subspace mapped to SL(2), we make use of results known for SL(2). It is known [6, p 156] that the complete set of irreducible sextic invariants for SL(2) consists of five polynomials. In the mathematical literature [6, 7], these are given in concise, yet symbolic, forms as follows:

$$\begin{aligned}&I_1=(f,f)^{(6)} \qquad I_2=(i,i)^{(4)}\\ &I_3=(l,l)^{(2)} \qquad I_4=(f,l^3)^{(6)}\\ &I_5=((f,i)^{(1)},l^4)^{(8)}\end{aligned} \tag{10}$$

where

$$\begin{aligned}&f\equiv(\alpha\cdot x)^6\\ &i\equiv(f,f)^{(4)}=(\alpha\beta)^4(\alpha\cdot x)^2(\beta\cdot x)^2\\ &l\equiv(f,i)^{(4)}=(\alpha\beta)^2(\alpha\gamma)^2(\beta\gamma)^4(\alpha\cdot x)^2\\ &\alpha\cdot x\equiv\alpha_1x_1+\alpha_2x_2 \qquad (\alpha\beta)\equiv\alpha_1\beta_2-\beta_1\alpha_2.\end{aligned} \tag{11}$$

Here, for any

$$\begin{aligned}f&=(\alpha\cdot x_1)(\alpha\cdot x_2)\ldots(\alpha\cdot x_m)\\ g&=(\beta\cdot x_1)(\beta\cdot x_2)\ldots(\beta\cdot x_n)\end{aligned} \tag{12}$$

we have

$$(f,g)^{(r)}\equiv C\sum_{P,Q}\frac{(\alpha_{P1}\beta_{Q1})(\alpha_{P2}\beta_{Q2})\ldots(\alpha_{Pr}\beta_{Qr})}{(\alpha\cdot x_{P1})(\alpha\cdot x_{P2})\ldots(\alpha\cdot x_{Pr})(\beta\cdot x_{Q1})(\beta\cdot x_{Q2})\ldots(\beta\cdot x_{Qr})}fg \tag{13}$$

where $C=[r!\binom{m}{r}\binom{n}{r}]^{-1}$ and the summation extends to all *distinct* permutations P and Q of the r integers $1, 2, \ldots, r$. In (10), the degree of the invariants as polynomials in the e_j is the same as the degree in the fs. Thus, we find I_1, I_2, I_3, I_4 and I_5 of degrees 2, 4, 6, 10 and 15, respectively, in the e_j.

We caution that the above notations are highly symbolic and should be deciphered with care. Particularly, since the αs have only symbolical meaning, they can be replaced by other symbols, i.e. $\alpha\cdot x=\beta\cdot x=\gamma\cdot x$. After some reductions, we find the following explicit expressions of fundamental invariants:

$$\begin{aligned}J_1&\equiv I_2/2=e_0e_6-6e_1e_5+15e_2e_4-10e_3^2\\ J_2&\equiv I_2/24-J_1^2/36=-e_3^4+e_3^2(e_0e_6+2e_1e_5+3e_2e_4)\\ &\quad+e_0e_4^3+e_0e_2e_5^2-2e_3e_4(e_1e_4+e_0e_5)+e_2e_4(2e_1e_5-e_0e_6)\\ &\quad+e_6e_2^3+e_6e_4e_1^2-2e_3e_2(e_5e_2+e_6e_1)-2e_1^2e_5^2.\end{aligned} \tag{14}$$

Explicit expressions of I_3, I_4 and I_5, which can be worked out in a straightforward fashion, are un-illuminatively complicated, and will not be presented. It may be explicitly verified by substituting (6) and (7) into (9) and (14) that the Is and Js are invariant under the permutation of the subscripts $\{i, j, k\}$ of the vertex weights ω_{ijk}, as required by the symmetry of the three spin states of the lattice edges.

Of special interest in statistical mechanical applications is the subspace $e_1 = e_3 = e_5 = 0$ pertaining to the spin-1 Blume–Emery–Griffiths model [15]. The intersections of the six fundamental invariants in this subspace possess a much simpler form. We find, in addition to I_0 and $I_5 \equiv 0$, the following expressions:

$$
\begin{aligned}
J_1 &= A + 15B \\
J_2 &= C - B^2 - AB \\
J_3 &= AC + 3BC - 2B^3 - 6AB^2 \\
J_4 &= 4(5A - 9B)C^2 + (A^3 + 21A^2B - 93AB^2 + 135B^3)C \\
&\qquad + 2B^2(9A^3 - 59A^2B + 99AB^2 - 81B^3)
\end{aligned} \tag{15}
$$

where $A = e_0e_6$, $B = e_2e_4$, $C = e_2^3e_6 + e_4^3e_0$, $J_3 = I_3/24 + 4J_1J_2/3$, $J_4 = I_4/64$.

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